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On bosonisation in (1+1) dimensions

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Abstract. In this paper we present the quantum field theory of the free scalar massless field and derive on this basis the operator solution to the Thirring model. The Lorentz and scale transformation formulae for the scalar and Thirring fields are found explicitly. It is shown that one can attribute definite spin and scale dimension to the Thirring field $\psi(x)$ only if one interprets $\psi(x)$ as an intertwining operator between inequivalent charged sectors. Particular attention is paid to the way the two charge operators are contained in the theory of the massless scalar field.

1. Introduction

The massless Thirring model for interacting relativistic fermions in two-dimensional space–time is exactly soluble, and several different approaches to the model have been introduced in the past. In 1977 Nakanishi found operator solutions to various two-dimensional models (Nakanishi 1977a, 1978) using his own formalism of the two-dimensional free massless scalar field (Nakanishi 1977b, 1980). As pointed out by Hadjiivanov *et al* (1979) and Hadjiivanov and Stoyanov (1979a) Nakanishi missed the second non-zero charge. This was the consequence of the asymptotic completeness of the single massless scalar field $\phi(x)$ combined with the assumed asymptotic behaviour of $\phi(x)$.

Both Nakanishi (1977b) and Hadjiivanov *et al* (1979) have investigated the transformation properties of the Thirring field $\psi(x)$ under Lorentz transformations. Hadjiivanov *et al* (1979) have also studied the transformation properties of $\psi(x)$ under scale transformations. This transformation cannot be introduced in Nakanishi's framework (see § 6). The transformation laws for the Thirring field obtained by these authors are not the standard ones; in particular one cannot assign any spin or scale dimension to the field $\psi(x)$. The reason for this is that the Thirring field acts in the indefinite metric space of a free scalar field. The main result of this paper is the explanation of how bosonisation in (1+1) dimensions is related to the association of a spin and scale dimension with the Fermi field.

It is well known (Nakanishi 1977c, Hadjiivanov and Stoyanov 1979b) that in the limit of $\mu_r \to 0$ the Wightman functions define the positive definite two-dimensional theory, in which gauge symmetries are restored and the Thirring field acquires fixed spin and scale dimension. We state that it is possible to quantise the massless scalar field theory in such a way that the Fock representation occurs in the Hilbert space with a positive metric. The two charges Φ and $\tilde{\Phi}$ define superselection sectors and the irreducible representations of commutation relations correspond to different charge sectors (Ezawa 1979, Streater 1971, 1973, 1974).

The Thirring field, which we reconstruct from the Bose fields, acts then as an intertwining operator from one charge sector to another.

The global gauge transformation is not spontaneously broken in the Fock representation in agreement with Coleman's theorem (Coleman 1973) which asserts the non-existence of Goldstone bosons in the positive two-dimensional theory. The transformation properties with respect to the conformal group will be studied in this paper in the canonical framework. The Bose constituents $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$ of the Thirring field $\psi(x)$ will transform inhomogeneously under Lorentz and scale transformations, in such a way that $\psi(x)$ transforms in the standard way under these transformations. These transformation laws are completely consistent with positivity conditions and, in particular, require the two-point functions including different components of the Thirring field (in a representation which diagonalises γ^5) to vanish.

Other authors (Ringwood 1979, Seiler and Uhlenbrock 1977, Freundlich 1972) used box normalisation to reveal the structure of the charge sectors and their relation to the fermion–boson correspondence.

Our formalism makes it possible to avoid the infrared divergencies, which arise in the theory of the scalar massless field in two dimensions, by the appropriate regularisation of some integrals and by restriction of the family of test functions in order to remove the boson zero-energy modes. In particular we do not need to introduce *ad hoc* the charge-raising operators (see Freundlich 1972) because we obtain, in a natural way, understanding of the Fermion field as an intertwining operator.

Hence there is no need to reformulate the two-dimensional theories by using a compact space formalism, which is not Lorentz invariant or by using the indefinite metric formalism which makes the fermion–boson equivalence purely formal.

The present paper is organised as follows: § 2 is devoted to a recapitulation of some elementary facts about solutions to the one-dimensional wave-equation. In § 3 we construct the parity conjugate field $\tilde{\phi}(x)$ and the charges Φ and $\tilde{\Phi}$. Particular attention is paid to the Poincaré and dilatation transformation properties of the fields $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$. In § 4 we present a heuristic approach to the construction of the charge raising operators and superselection rules for the free massless boson field.

The Thirring model is discussed in § 5. We emphasise the fact that one can attribute definite spin and scale dimension to the Thirring field $\psi(x)$ only if one interprets $\psi(x)$ as an intertwining operator between inequivalent charged sectors. In § 6 we make several remarks about the formalism presented by Nakanishi and Hadjiivanov *et al.* We reproduce Hadjiivanov's transformation laws from the canonical approach and show that they can be given an interpretation consistent with our conclusions.

In the appendix, the expressions for $D^{(\pm)}(x)$ and $\tilde{D}^{(\pm)}(x)$ and related formulae are summarised. Throughout this paper we follow the conventions of Nakanishi (1977a, 1978, 1980).

2. The parity conjugate of massless fields and commutator functions $D(x)$ and $\tilde{D}(x)$

The one-dimensional wave-equation $\Box f(x)=0$ is satisfied if and only if $f(x)$ is of the form

$$f(x_0, x_1)=f_R(x_0-x_1)+f_L(x_0+x_1)$$

where f_R and f_L are any differentiable functions of one variable. The variables x_0-x_1

and x_0+x_1 are used in the light cone formulation of the Thirring model (Dell-Antonio 1972).

Instead of working with f_R and f_L we can take another approach. Let us introduce two invariant solutions of the wave-equation

$$D(x)=\frac{1}{2\pi \mathrm{i}}\int \mathrm{d}^2p\, \varepsilon(p^0)\,\delta(p^2)\,\mathrm{e}^{-\mathrm{i}px}=-\tfrac{1}{2}\varepsilon(x^0)\theta(x^2)=-\tfrac{1}{2}[(\theta(x^1+x^0)-\theta(x^1-x^0)] \quad (2.1)$$

$$\tilde{D}(x)=-\frac{1}{2\pi \mathrm{i}}\int \mathrm{d}^2p\, \varepsilon(p^1)\,\delta(p^2)\,\mathrm{e}^{-\mathrm{i}px}=-\tfrac{1}{2}\varepsilon(x^1)\theta(-x^2)=-\tfrac{1}{2}[\theta(x^0+x^1)-\theta(x^0-x^1)]. \quad (2.2)$$

The existence of $\tilde{D}(x)$ is intimately connected with the two-dimensionality. $D(x)$ and $\tilde{D}(x)$ are related by

$$\partial_\mu D(x)+\varepsilon_{\mu\nu}\partial^\nu \tilde{D}(x)=0. \quad (2.3)$$

If $f(x)$ is any function satisfying $\Box f(x)=0$, then (Nakanishi 1977b)

$$f(x)=\int_{-\infty}^{\infty}\mathrm{d}y^1\, f(y)\overleftrightarrow{\partial}_0 D(y-x) \quad (2.4)$$

where

$$f\overleftrightarrow{\partial}_0 g=\left(\frac{\partial f}{\partial y_0}\right)g-f\frac{\partial g}{\partial y_0}.$$

Now, when $f(x)$ satisfies the massless equation, there is a function $\tilde{f}(x)$ which is related to $f(x)$ by

$$\tilde{f}(x)=\int_{-\infty}^{\infty}\mathrm{d}y^1 f(y)\overleftrightarrow{\partial}_0\tilde{D}(y-x). \quad (2.5)$$

By equation (2.3) this relation can be written in differential form

$$\partial_\mu f(x)+\varepsilon_{\mu\nu}\partial^\nu\tilde{f}(x)=0 \quad (2.6)$$

or

$$\partial_\mu\tilde{f}(x)+\varepsilon_{\mu\nu}\partial^\nu f(x)=0$$

and $\Box\tilde{f}(x)=0$. $\tilde{f}(x)$ us called the parity conjugate of $f(x)$, as $f(x)$ and $\tilde{f}(x)$ cannot simultaneously be of the same parity. When $\partial_1 f(x)$ vanishes at infinity we can show that the right-hand side of (2.5) is independent of y_0. Then we set $y_0=x_0$ and obtain

$$\tilde{f}(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y^1\,\varepsilon(x^1-y^1)\partial_0 f(x^0,y^1). \quad (2.7)$$

For consistency between equations (2.6) and (2.7) it is necessary to assume $\tilde{f}(x^0,+\infty)+\tilde{f}(x^0,-\infty)=0$.

Now we can also write (2.6) in integral form as

$$f(x)=\int_{-\infty}^{\infty}\mathrm{d}y^1\,\tilde{f}(y)\overleftrightarrow{\partial}_0\tilde{D}(y-x) \quad (2.8)$$

or

$$f(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y^1\,\varepsilon(x^1-y^1)\partial_0\tilde{f}(x^0,y^1) \quad (2.9)$$

where we have assumed that $f(x^0, +\infty)+f(x^0, -\infty)=0$. As an example we note that $D(x)$, $\tilde{D}(x)$ and $\tilde{D}^{(\pm)}(x)$ satisfy the condition

$$f(x^0, +\infty)+f(x^0, -\infty)=0$$

(e.g. $\lim_{x^1\to\pm\infty}\tilde{D}^{(\pm)}(x) = \mp i/4$) and the spatial derivatives of these three functions vanish at infinity. Equations (2.5) and (2.7) yield

$$\tilde{D}(x)=\int_{-\infty}^{\infty} dy^1\, D(y)\bar{\partial}_0\tilde{D}(y-x)=\tfrac{1}{2}\int_{-\infty}^{\infty} dy^1\, \varepsilon(x^1-y^1)\partial_0 D(x^0, y^1)$$

$$D(x)=\int_{-\infty}^{\infty} dy^1\, \tilde{D}(y)\bar{\partial}_0\tilde{D}(y-x)=\tfrac{1}{2}\int_{-\infty}^{\infty} dy^1\, \varepsilon(x^1-y^1)\partial_0 \tilde{D}(x^0, y^1)$$

$$\tilde{D}^{(\pm)}(x)=\int_{-\infty}^{\infty} dy^1\, D^{(\pm)}(y)\bar{\partial}_0\tilde{D}(y-x)=\tfrac{1}{2}\int_{-\infty}^{\infty} dy^1\, \varepsilon(x^1-y^1)\partial_0 D^{(\pm)}(x^0, y^1).$$

We introduce the Fourier transform of any $f(x)$ which satisfies the massless wave-equation

$$f(x)=f_p(x)+f_n(x)=\int_{-\infty}^{\infty}\frac{dp^1}{2p^0}(\hat{f}_p(p^1)\, e^{-ipx}+\hat{f}_n(p^1)\, e^{ipx}) \tag{2.11}$$

where p means non-negative and n non-positive frequency solutions to $\Box f(x)=0$. It is understood in (2.11) that for $\hat{f}_p(0)=\hat{f}_n(0)=0$. We now define $f^{(\pm)}(x)$ as

$$f^{(\pm)}(x)=-i\int_{-\infty}^{\infty} dy^1\, D^{(\pm)}(x-y)\overleftrightarrow{\partial}_0 f(y)=-i\int_{-\infty}^{\infty} dy^1\, f(y)\overleftrightarrow{\partial}_0 D^{(\mp)}(y-x). \tag{2.12}$$

It is straightforward to show that

$$f^{(+)}(x)=f_p(x) \qquad f^{(-)}(x)=f_n(x) \tag{2.13}$$

and

$$f_n^{(+)}(x)=0 \qquad f_p^{(-)}=0. \tag{2.14}$$

3. The scalar massless field

The scalar zero-mass quantum field in two space–time dimensions plays a fundamental role in the construction of solutions to solvable two-dimensional models. As we shall see, the original features of this theory lead directly to the well known Lorentz and dilatation transformation properties of the Thirring field.

The field equation and the commutation relation for the quantum scalar field $\phi(x)$ in two dimensions are

$$\Box\phi(x)=0 \tag{3.1}$$

$$[\phi(x), \phi(y)]=iD(x-y). \tag{3.2}$$

There exists a conjugate field $\tilde{\phi}(x)$ of $\phi(x)$, which is related to $\phi(x)$ by

$$\partial_\mu\phi(x)+\varepsilon_{\mu\nu}\,\partial^\nu\tilde{\phi}(x)=0 \tag{3.3}$$

and satisfies of course the wave-equation

$$\Box\tilde{\phi}(x)=0.$$

Because we work in Langrangian framework with

$$\mathscr{L}=\tfrac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi=-\tfrac{1}{2}\partial_{\mu}\tilde{\phi}\partial^{\mu}\tilde{\phi}$$

we must require for the conjugate field the same commutation relation as for $\phi(x)$

$$[\tilde{\phi}(x),\tilde{\phi}(y)]=\mathrm{i}D(x-y). \tag{3.4}$$

Instead of defining a conjugate field by the integral which is a copy of equation (2.7),

$$\tilde{\phi}(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y^{1}\,\varepsilon(x^{1}-y^{1})\partial_{0}\phi(x^{0},y^{1}) \tag{3.5}$$

with possible (and really existing, as we shall see) convergence problems at the lower and upper limit of integration, we can guess directly the commutation relation between $\phi(x)$ and $\tilde{\phi}(x)$.

Using equations (3.3) and (3.4) as a definition of the field $\tilde{\phi}(x)$ we find

$$[\phi(x),\tilde{\phi}(y)]=\mathrm{i}\tilde{D}(x-y)+c \tag{3.6}$$

where c is an imaginary number. Parity considerations (from (3.3) we see that $\phi(x)$ and $\tilde{\phi}(x)$ cannot simultaneously be of the same parity) show that the constant term c must be excluded from (3.6). With $c=0$, equation (3.6) (like equation (3.4)) is symmetric under the interchanging $\phi\leftrightarrow\tilde{\phi}$. When the Fock space is endowed with a positive metric (see § 4), the constant c will in any case play no role. The commutation relation

$$[\phi(x),\tilde{\phi}(y)]=\mathrm{i}\tilde{D}(x-y) \tag{3.7}$$

makes it possible to consider both $\phi(x)$ and $\tilde{\phi}(x)$ as fundamental objects. It is clear that (3.7) and (3.4) are the immediate consequences of (3.5) together with (3.2) and the definition

$$\phi(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y^{1}\,\varepsilon(x^{1}-y^{1})\partial_{0}\tilde{\phi}(x^{0},y^{1}) \tag{3.8}$$

leads from equation (3.4) to (3.2) and (3.7).

Equation (3.8) shows that $\phi(x)$ can be expressed in terms of $\tilde{\phi}(x)$ and $\tilde{\phi}(x)$ can be expressed in terms of $\phi(x)$ using (3.5).

As we emphasised above these two equations cannot be used as the definition of $\tilde{\phi}(x)$ or $\phi(x)$ owing to their convergence problems (see (3.13)).

In order to separate two frequency parts of the fields $\phi(x)$ and $\tilde{\phi}(x)$ we use, following Nakanishi (1977b), the formula

$$\phi^{(\pm)}(x)=-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}y^{1}\,D^{(\pm)}(x-y)\overleftrightarrow{\partial}_{0}\phi(y) \tag{3.9}$$

and

$$\tilde{\phi}^{(\pm)}(x)=-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}y^{1}\,D^{(\pm)}(x-y)\overleftrightarrow{\partial}_{0}\tilde{\phi}(y). \tag{3.10}$$

From these definitions, (3.2), (3.4) and (3.7) and with the help of (A.6)–(A.9) we find that the only non-vanishing commutators are

$$[\phi^{(\pm)}(x),\phi^{(\mp)}(y)]=[\tilde{\phi}^{(\pm)}(x),\tilde{\phi}^{(\mp)}(y)]=D^{(\pm)}(x-y) \tag{3.11}$$

$$[\phi^{(\pm)}(x)_{1}\tilde{\phi}^{(\mp)}(y)]=\tilde{D}^{(\pm)}(x-y). \tag{3.12}$$

Because $D^{(\pm)}(x)$ is divergent as $|x^1|\to\infty$ we cannot use equation (3.5) in (3.11) and (3.12); explicitly

$$D^{(+)}(x-y)=[\tilde{\phi}^{(+)}(x)_1\tilde{\phi}(y)]$$
$$=\left(\tilde{\phi}^{(+)}(x),\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}z^1\,\varepsilon(y^1-z^1)\partial_0\phi(y^0,z^1)\right)$$
$$=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}z^1\,\varepsilon(y^1-z^1)\partial_0\tilde{D}^{(+)}(x^0-y^0,x^1-z^1)$$
$$=D^{(+)}(x-y)-\tfrac{1}{2}[D^{(+)}(x^0-y^0,+\infty)+D^{(+)}(x^0-y^0,-\infty)]. \tag{3.13}$$

In other words, it is impossible to define $D^{(\pm)}(x)$ as

$$\int_{-\infty}^{\infty}\mathrm{d}y^1\,\tilde{D}^{(\pm)}(y)\tilde{\partial}_0\tilde{D}(y-x).$$

We next consider the space integrals

$$\Phi_L=\int_{-L}^{L}\mathrm{d}x^1\,\partial_0\phi(x)$$

and

$$\tilde{\Phi}_L=\int_{-L}^{L}\mathrm{d}x^1\,\partial_0\tilde{\phi}(x).$$

Although the limit of Φ_L and $\tilde{\Phi}_L$ at $L\to\infty$ does not exist, there are no difficulties with the limit in

$$\int_{-L}^{L}\mathrm{d}x^1[\partial_0\phi(x),\cdot] \tag{3.14}$$

and

$$\int_{-L}^{L}\mathrm{d}x^1[\partial_0\tilde{\phi}(x),\cdot] \tag{3.15}$$

where $\cdot$ stands for the operators $\phi(x)$, $\tilde{\phi}(x)$, $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$. As an illustration we calculate the following commutation relations

$$[\Phi_\infty,\phi(x)]=\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}y^1\,\partial_0 D(y-x)=-\mathrm{i}$$
$$[\Phi_\infty,\phi^{(\pm)}(x)]=\int_{-\infty}^{\infty}\mathrm{d}y^1\,\partial_0 D^{(\mp)}(y-x)=-\frac{\mathrm{i}}{2}$$
$$[\Phi_\infty,\tilde{\phi}(x)]=\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}y^1\,\partial_0\tilde{D}(y-x)=0.$$

However, there is a difficulty with

$$[\Phi_\infty,\tilde{\phi}^{(\pm)}(x)]=\int_{-\infty}^{\infty}\mathrm{d}y^1\,\partial_0\tilde{D}^{(\mp)}(y-x)=D^{(\mp)}(+\infty)-D^{(\mp)}(-\infty)$$

and we must calculate this in the following way

$$\lim_{L\to\infty} [\Phi_L, \tilde{\phi}^{(\pm)}(x)] = \lim_{L\to\infty}[D^{(\mp)}(L) - D^{(\mp)}(-L)] = 0. \tag{3.16}$$

We can now define two important quantities

$$\Phi = \lim_{L\to\infty} \int_{-L}^{L} dx^1\, \partial_0\phi(x) \tag{3.17}$$

$$\tilde{\Phi} = \lim_{L\to\infty} \int_{-L}^{L} dx^1\, \partial_0\tilde{\phi}(x) \tag{3.18}$$

where $\lim_{L\to\infty} \int_{-L}^{L} dx^1$ means regularisation as described above in (3.15). It is now clear that we have the following commutation relations

$$[\Phi, \phi(x)] = -i \qquad [\Phi, \phi^{(\pm)}(x)] = -\tfrac{1}{2}i \tag{3.19}$$

and similarly

$$[\tilde{\Phi}, \tilde{\phi}(x)] = -i \qquad [\tilde{\Phi}, \tilde{\phi}^{(\pm)}(x)] = -\tfrac{1}{2}i. \tag{3.20}$$

It is easy to see that Φ commutes with $\tilde{\phi}^{(\pm)}(x)$ and $\tilde{\Phi}$ with $\phi^{(\pm)}(x)$.

It is also possible to separate Φ and $\tilde{\Phi}$ into two frequency parts

$$\Phi^{(\pm)} = \lim_{L\to\infty} \int_{-L}^{L} dx^1\, \partial_0\phi^{(\pm)}(x) \tag{3.21}$$

$$\tilde{\Phi}^{(\pm)} = \lim_{L\to\infty} \int_{-L}^{L} dx^1\, \partial_0\tilde{\phi}^{(\pm)}(x). \tag{3.22}$$

All these operators defined in (3.17), (3.18), (3.21) and (3.22) commute with each other. The convergence problems with expressions (3.5) and (3.8) shown explicitly in (3.13) suggest that, in contrast with the usual theory, $\lim_{x^1\to\pm\infty} \phi(x)$ does not vanish. But we can assume without contradicting the commutation relations (3.11)–(3.12) that

$$\lim_{|x^1|\to\infty} (x^1)^{-\varepsilon}\phi(x) = \lim_{|x^1|\to\infty} (x^1)^{-\varepsilon}\tilde{\phi}(x) = 0 \qquad \text{for any } \varepsilon > 0. \tag{3.23}$$

To ensure convergence in definitions (3.9) and (3.10) we must regulate the integrals

$$\phi^{(\pm)}(x) = -i \lim_{L\to\infty} \int_{-L}^{L} dy^1\, D^{(\pm)}(x-y)\overleftrightarrow{\partial}_0\phi(y) \tag{3.24}$$

$$\tilde{\phi}^{(\pm)}(x) = i \lim_{L\to\infty} \int_{-L}^{L} dy^1\, D^{(\pm)}(x-y)\overleftrightarrow{\partial}_0\tilde{\phi}(y). \tag{3.25}$$

Postulate (3.23) ensures that $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$ are conjugate to each other

$$\partial_\mu\phi^{(\pm)}(x) + \varepsilon_{\mu\nu}\partial^\nu\tilde{\phi}^{(\pm)}(x) = 0. \tag{3.26}$$

Our next goal is to investigate the behaviour of a free boson field under Lorentz and scale transformations. The energy–momentum tensor

$$\Theta^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \tfrac{1}{2}g^{\mu\nu}\partial^\rho\phi\partial_\rho\phi \tag{3.27}$$

is symmetric, conserved and traceless. Therefore the conserved Poincaré generators and dilatation charge can be constructed and are given by

$$P_0 \equiv \tfrac{1}{2}\int_{-\infty}^{\infty} dx^1[(\partial_0\phi)^2+(\partial_1\phi)^2] \tag{3.28}$$

$$P_1 \equiv \int_{-\infty}^{\infty} dx^1\,\partial_0\phi\,\partial_1\phi \tag{3.29}$$

$$M_{01} \equiv \lim_{L\to\infty}\int_{-L}^{L} dx^1\{x_0\partial_0\phi\,\partial_1\phi-\tfrac{1}{2}x_1[(\partial_0\phi)^2+(\partial_1\phi)^2]\} \tag{3.30}$$

$$D \equiv \lim_{L\to\infty}\int_{-L}^{L} dx^1\{x_1\partial_0\phi\,\partial_1\phi-\tfrac{1}{2}x_0[(\partial_0\phi)^2+(\partial_1\phi)^2]\}. \tag{3.31}$$

Since $\partial_\mu\phi(x)+\varepsilon_{\mu\nu}\partial^\nu\tilde{\phi}(x)=0$, these generators can be expressed in terms of $\tilde{\phi}(x)$ in the same form as above.

Let us assume for the moment that these free scalar fields $\phi(x)$ and $\tilde{\phi}(x)$ transform canonically under Poincaré transformations:

$$[\phi(x), M_{01}] = \mathrm{i}(x_0\partial_1-x_1\partial_0)\phi(x) \tag{3.32}$$

$$[\tilde{\phi}(x), M_{01}] = \mathrm{i}(x_0\partial_1-x_1\partial_0)\tilde{\phi}(x). \tag{3.33}$$

Now, the definition of $\phi^{(\pm)}(x)$ given in (3.24) implies

$$[\phi^{(\pm)}(x), M_{01}] = \lim_{L\to\infty}\int_{-L}^{L} dy^1\,D^{(\pm)}(x-y)\overleftrightarrow{\partial}_0(y_0\partial_1-y_1\partial_0)\phi(y). \tag{3.34}$$

By performing an integration by parts and using $\partial_0^2\phi(x)=\partial_1^2\phi(x)$ the right-hand side of (3.34) becomes

$$\lim_{L\to\infty}[D^{(\pm)}(x-y)(y_1\partial_1-y_0\partial_0)\phi(y)]_{y^1=-L}^{y^1=L}$$

$$+\lim_{L\to\infty}\int_{-L}^{L} dy^1(y_0\partial_0^y-y_1\partial_1^y)D^{(\pm)}(x-y)\partial_1\phi(y)$$

$$+\lim_{L\to\infty}\int_{-L}^{L} dy^1(y_0\partial_1^y-y_1\partial_0^y)D^{(\pm)}(x-y)\partial_0\phi(y). \tag{3.35}$$

The surface term in (3.35) vanishes by the same reasoning as in (3.16). If we now make use of the formulae presented in the Appendix (A.15)–(A.16) we obtain

$$[\phi^{(\pm)}(x), M_{01}]$$

$$= \lim_{L\to\infty}\int_{-L}^{L} dy^1(x_1\partial_1-x_0\partial_0)D^{(\pm)}(x-y)\partial_1\phi(y) \mp \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L} dy^1\,\partial_1\phi(y)$$

$$+\lim_{L\to\infty}\int_{-L}^{L} dy^1(x_1\partial_0-x_0\partial_1)D^{(\pm)}(x-y)\partial_0\phi(y). \tag{3.36}$$

Integrating by parts the first term on the right-hand side of (3.36) we obtain

$$[\phi^{(\pm)}(x), M_{01}] = \mathrm{i}(x_0\partial_1-x_1\partial_0)\phi^{(\pm)}(x)\mp\Phi/2\pi. \tag{3.37}$$

In the same way as before we obtain

$$[\tilde{\phi}^{(\pm)}(x), M_{01}] = \mathrm{i}(x_0\partial_1-x_1\partial_0)\tilde{\phi}^{(\pm)}(x)\mp\tilde{\Phi}/2\pi. \tag{3.38}$$

i.e. $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$ are transformed under the Lorentz transformation

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix}$$

as

$$\phi^{(\pm)\prime}(x') = \phi^{(\pm)}(x) \pm i\chi\tilde{\Phi}/2\pi \tag{3.39}$$

$$\phi^{(\pm)\prime}(x') = \tilde{\phi}^{(\pm)}(x) \pm i\chi\Phi/2\pi \tag{3.40}$$

where

$$x'^{\mu} = (\Lambda^{-1})^{\mu}\nu x^{\nu}.$$

Assuming that the free fields $\phi(x)$, $\tilde{\phi}(x)$ transform canonically under scale transformations

$$[\phi(x), D] = -ix^{\mu}\partial_{\mu}\phi(x) \tag{3.41}$$

$$[\tilde{\phi}(x), D] = -ix^{\mu}\partial_{\mu}\tilde{\phi}(x) \tag{3.42}$$

and repeating the same considerations as above we obtain

$$[\phi^{(\pm)}(x), D] = -ix^{\mu}\partial_{\mu}\phi^{(\pm)}(x) \pm \Phi/2\pi \tag{3.43}$$

or

$$\phi^{(\pm)\prime}(x) = \phi^{(\pm)}(\lambda x) \pm i \ln \lambda\, \Phi/2\pi \tag{3.44}$$

and the same relations with $\tilde{\phi}^{(\pm)}(x)$ and $\tilde{\Phi}$ instead of $\phi^{(\pm)}(x)$ and Φ. If the scalar fields $\phi(x)$ and $\tilde{\phi}(x)$ transform inhomogeneously under Lorentz transformations we expect that

$$[\phi(x), M_{01}] = i(x_0\partial_1 - x_1\partial_0)\phi(x) + F \tag{3.45}$$

$$[\tilde{\phi}(x), M_{01}] = i(x_0\partial_1 - x_1\partial_o)\tilde{\phi}(x) + G \tag{3.46}$$

where F and G are some constant operators.

If we insert equations (3.39)–(3.40) in our commutation relations (3.11)–(3.12) then it is easy to see that they are consistent with the properties of $D^{(\pm)}(x)$ and $\tilde{D}^{(\pm)}(x)$ under Lorentz transformations (see the Appendix). Therefore the extra terms F and G should give no contribution to the commutation relations. Then the only possible choice for F and G is

$$F = k\tilde{\Phi} \qquad \text{and} \qquad G = k\Phi$$

where k is some arbitrary number.

If we modify equations (3.41)–(3.42) by

$$[\phi(x), D] = -ix_{\mu}\partial^{\mu}\phi(x) + F$$

$$[\tilde{\phi}(x), D] = -ix_{\mu}\partial^{\mu}\tilde{\phi}(x) + G$$

then the same considerations as above restrict F and G to be $k_1\Phi$ and $k_s\tilde{\Phi}$ respectively.

From (3.28)–(3.29) we derive directly

$$[\phi^{(\pm)}(x), P_{\mu}] = i\partial_{\mu}\phi^{(\pm)}(x) \tag{3.47}$$

$$[\tilde{\phi}^{(\pm)}(x), P_{\mu}] = i\partial_{\mu}\tilde{\phi}^{(\pm)}(x) \tag{3.48}$$

$$[\phi^{(\pm)}(x), M_{01}] = i(x_0\partial_1 - x_1\partial_0)\phi^{(\pm)}(x) \mp \tilde{\Phi}/2\pi \tag{3.49}$$

$$[\tilde{\phi}^{(\pm)}(x), M_{01}] = \mathrm{i}(x_0\partial_1 - x_1\partial_0)\tilde{\phi}^{(\pm)}(x) \mp \Phi/2\pi \tag{3.50}$$

$$[\phi^{(\pm)}(x), D] = -\mathrm{i}x^\mu\partial_\mu\phi^{(\pm)}(x) \pm \Phi/2\pi \tag{3.51}$$

$$[\tilde{\phi}^{(\pm)}(x), D] = -\mathrm{i}x^\mu\partial_\mu\tilde{\phi}^{(\pm)}(x) \pm \tilde{\Phi}/2\pi. \tag{3.52}$$

As an example we prove (3.49), by using commutation relations (3.11)–(3.12). We obtain

$$\begin{aligned}
[\phi^{(\pm)}(x), M_{01}]& \\
&= \lim_{L\to\infty}\int_{-L}^{L} \mathrm{d}y^1[y_0(\partial_0^y D^{(\pm)}(x-y)\partial_1\phi(y) + \partial_1^y D^{(\pm)}(x-y)\partial_0\phi(y)) \\
&\quad - y_1(\partial_0^y D^{(\pm)}(x-y)\partial_0\phi(y) + \partial_1^y D^{(\pm)}(x-y)\partial_1\phi(y))] \\
&= -\lim_{L\to\infty}\int_{-L}^{L} \mathrm{d}y^1[(y_0\partial_0 - y_1\partial_1)D^{(\mp)}(y-x)\partial_1\phi(y) \\
&\quad + (y_0\partial_1 - y_1\partial_0)D^{(\mp)}(y-x)\partial_0\phi(y)].
\end{aligned}$$

According to equations (A.15)–(A.16) we find

$$\begin{aligned}
[\phi^{(\pm)}(x), M_{01}]& \\
&= -\lim_{L\to\infty}\int_{-L}^{L} \mathrm{d}y^1\left[\left((x_0\partial_0 - x_1\partial_1)D^{(\mp)}(y-x) \pm \frac{1}{2\pi}\right)\partial_1\phi(y)\right. \\
&\quad \left. + (x_0\partial_1 - x_1\partial_0)D^{(\mp)}(y-x)\partial_0\phi(y)\right].
\end{aligned}$$

By using $(\partial_0)^2D^{(\pm)}(x) = (\partial_1)^2D^{(\pm)}(x)$ and integrating by parts in y_1, the last equation becomes

$$\begin{aligned}
[\phi^{(\pm)}(x), M_{01}] &= -(x_0\partial_1 - x_1\partial_0)\lim_{L\to\infty}\int_{-L}^{L} \mathrm{d}y^1\left[\partial_0 D^{(\mp)}(y-x)\phi(y)\right. \\
&\quad \left. - D^{(\mp)}(y-x)\partial_0\phi(y) \mp \frac{1}{2\pi}\partial_1\phi(y)\right] \\
&= \mathrm{i}(x_0\partial_1 - x_1\partial_0)\phi^{(\pm)}(x) \mp \tilde{\Phi}/2\pi.
\end{aligned}$$

Then the right-hand side of (3.49) is obtained. From (3.47)–(3.52) we conclude that the free scalar fields $\phi(x)$ and $\tilde{\phi}(x)$ transform canonically under Poincaré and scale transformations. We see that formulae (3.47)–(3.52) agree with those we found previously using only the definitions of $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$, and that they leave all commutators of the scalar fields (equal frequency commutators included) invariant.

4. Superselection rules of a free massless boson field

It is well known that in order to work with the positive definite metric it is necessary to restrict the class of the test functions allowed for $\phi(x)$. Let us consider an indefinite Hermitian form

$$\langle f|g\rangle = \mathrm{i}\int \mathrm{d}x^1\, g^{(+)}(x)\overleftrightarrow{\partial}_0 f^{*(-)}(x) \tag{4.1}$$

with functions $f(x)$ and $g(x)$ obeying

$$\Box f(x)=\Box g(x)=0.$$

Now, if we insert in (4.1) the non-negative frequency solution to $\Box f(x)=0$,

$$f(x)=\int_{-\infty}^{\infty}\frac{dp^1}{2p^0}\hat{f}_p(p^1)\,e^{-ipx} \tag{4.2}$$

with the condition

$$\hat{f}_p(0)=0 \tag{4.3}$$

then the form becomes positive definite and

$$\langle f|f\rangle=\pi\int_{-\infty}^{\infty}\frac{dp^1}{p^0}|\hat{f}(p^1)|^2>0 \tag{4.4}$$

when $\hat{f}_p(p^1)\neq 0$.

We can now introduce the Fock space F by identifying the form $\langle\cdot|\cdot\rangle$ as the scalar product of two states. For this purpose we define the vacuum $|0\}$ by

$$\int_{-\infty}^{\infty}dx^1\,f(x)\overleftrightarrow{\partial}_0\phi^{(+)}(x)|0\}=\int_{-\infty}^{\infty}dx^1\,f(x)\overleftrightarrow{\partial}_0\tilde{\phi}^{(+)}(x)|0\}=0 \tag{4.5}$$

with $f(x)$ defined in (2.11). If

$$|f\rangle=\int_{-\infty}^{\infty}dx^1\,f(x)\overleftrightarrow{\partial}_0\phi^{(-)}(x)|0\} \tag{4.6}$$

with $f(x)$ now given by (4.2), is the one-particle state then $\langle f|g\rangle$ is just reproduced as the scalar product of two states $|f\rangle$ and $|g\rangle$. If

$$|f\rangle^{\sim}=\int_{-\infty}^{\infty}dx^1\,f(x)\overleftrightarrow{\partial}_0\tilde{\phi}^{(-)}(x)|0\} \tag{4.7}$$

then

$$\langle f|g\rangle={}^{\sim}\langle f|g\rangle^{\sim}. \tag{4.8}$$

In order to introduce Φ and $\tilde{\Phi}$ on the Fock space F we rewrite definition (4.2) as (Nakanishi 1977a, 1978, 1980)

$$f_1(x)=\int_{-\infty}^{\infty}\frac{dp^1}{2p^0}\hat{f}_p(p^1)\,e^{-ipx}-c=f(x)-c \tag{4.9}$$

where $\hat{f}_p(p^1)$ still obeys (4.3) and c is a constant number. It is straightforward to show that we still have

$$\langle f_1|f_1\rangle=\pi\int_{-\infty}^{\infty}\frac{dp^1}{p^0}|\hat{f}_p(p^1)|^2$$

by using the fact that $c\int_{-\infty}^{\infty}dx^1\,\partial_0 f(x)=-ic\pi\hat{f}_p(0)=0$. This means that

$$\langle f_1|c\rangle=0 \tag{4.10}$$

where

$$|c\rangle=c\int_{-\infty}^{\infty}dx^1\,\partial_0\phi^{(-)}(x)|0\}=c\Phi^{(-)}|0\}. \tag{4.11}$$

By substituting c in (4.5) we obtain

$$\Phi^{(+)}|0\} = \tilde{\Phi}^{(+)}|0\} = 0. \tag{4.12}$$

We see that $c\Phi^{(-)}|0\}$ and $c\Phi^{(-)}|0\}$ form the zero norm states and must be orthogonal to any one-particle state on the Fock space. We check this directly:

$$\langle c|f\rangle = \left\{0|c\Phi^{(+)}\int_{-\infty}^{\infty} dx^1\, f(x)\overleftrightarrow{\partial}_0\phi^{(-)}(x)|0\right\} = c\int_{-\infty}^{\infty} dx^1\, \partial_0 f(x) = 0$$

$$\langle c|f\rangle^{\sim} = \{0|c\Phi^{(+)}\int_{-\infty}^{\infty} dx^1\, f(x)\overleftrightarrow{\partial}_0\tilde{\phi}^{(-)}(x)|0\}$$

$$= \lim_{L\to\infty} c\int_{-L}^{L} dy^1\, \partial_0^y \int_{-\infty}^{\infty} dx^1\, f(x)\overleftrightarrow{\partial}_0\tilde{D}^{(+)}(x-y) = -\mathrm{i}c \lim_{L\to\infty}\int_{-L}^{L} dy^1\, \partial_0\tilde{f}(y). \tag{4.14}$$

Thus we must additionally postulate

$$\int_{-\infty}^{\infty} dx^1\, \partial_0\tilde{f}(x) = f(\infty) - f(-\infty) = 0. \tag{4.15}$$

From the discussion of § 1 we know that

$$f(\infty) + f(-\infty) = 0 \qquad \tilde{f}(\infty) + \tilde{f}(-\infty) = 0.$$

Hence

$$f(\infty) = f(-\infty) = 0 \qquad \tilde{f}(\infty) = \tilde{f}(-\infty) = 0. \tag{4.16}$$

Since the zero-norm states in F are orthogonal to any vector in F (we assume that all test functions satisfy conditions (4.16)) we can identify the physical space, as usual, with the quotient space $H_{0,0} = H/H_0$, where H_0 is a subspace of a zero-norm state and H is the total Fock space given by

$$H = \sum_{n=1}^{\infty} \oplus [\otimes F]^n. \tag{4.17}$$

The matrix elements of physical fields in H depend only on equivalence classes, i.e. on vectors in $H_{0,0}$. If $|0'\}$ is an equivalence class in $H_{0,0}$ which contains $|0\}$ from H, then

$$\Phi|0'\} = \tilde{\Phi}|0'\} = 0 \tag{4.18}$$

i.e. both charges Φ and $\tilde{\Phi}$ annihilate the new vacuum. The operators Φ and $\tilde{\Phi}$ commute with all local, physical field operators from H, as

$$\phi^{(\pm)}(f) \equiv \int_{-\infty}^{\infty} dx^1\, f(x)\overleftrightarrow{\partial}_0\phi^{(\pm)}(x) \qquad \text{and} \qquad \tilde{\phi}^{(\pm)}(f) = \int dx^1\, f(x)\overleftrightarrow{\partial}_0\tilde{\phi}^{(\pm)}(x)$$

with $f(x)$ satisfying (4.16). Hence Φ and $\tilde{\Phi}$ define superselection rules with $H_{0,0}$ as a zero sector. We must additionally postulate the vacuum state to be invariant under Poincaré and dilatation transformations. This assumption is consistent with the conditions (4.18) because Φ and $\tilde{\Phi}$ are Poincaré—and scale—invariant. Moreover it is easy to see that if the function $f(x)$ obeys two conditions (4.3) and (4.15), the functions $f_{(\Lambda,a)}(x) \equiv f(\Lambda x + a)$ and $f_\lambda(x) \equiv f(\lambda x)$ would obey them too. In conclusion, the vacuum state $|0\}$ defined by (4.5) is invariant under these two transformations.

The Lagrangian $\mathscr{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is invariant under global gauge transformation

$$\phi(x)\to\phi(x)+a \qquad \text{where} \qquad a\in \boldsymbol{R}. \tag{4.19}$$

It is easy to prove that this transformation is not unitarily implementable in four dimensions (in the Fock representation). But in our case the charge Φ generating the symmetry vanishes on the physical space and this forces the symmetry to act trivially (as the identity) on $H_{0,0}$, i.e.

$$\int_{-\infty}^{\infty}\mathrm{d}x^1 f(x)\overleftrightarrow{\partial}_0(\phi(x)+a)=\int_{-\infty}^{\infty}\mathrm{d}x^1 f(x)\overleftrightarrow{\partial}_0\phi(x).$$

The same remark can be made about the symmetry

$$\tilde{\phi}(x)\to\tilde{\phi}(x)+b. \tag{4.20}$$

We have identified the Fock representation with the charge zero sector. The different charge sectors will now be shown to correspond to inequivalent representations. For this purpose we shall study the field translations

$$\phi^{(\pm)}(x)\to\phi_{\mathrm{f}}^{(\pm)}(x)=\phi^{(\pm)}(x)+f^{(\pm)}(x) \tag{4.21a}$$

$$\tilde{\phi}^{(\pm)}(x)\to\tilde{\phi}_{\mathrm{f}}^{(\pm)}(x)=\tilde{\phi}^{(\pm)}(x)+\tilde{f}^{(\pm)}(x). \tag{4.21b}$$

If we assume the conditions

$$\int_{-\infty}^{\infty}\mathrm{d}x^1\,\partial_0 f(x)=\alpha\neq 0 \tag{4.22}$$

$$\int_{-\infty}^{\infty}\mathrm{d}x^1\,\partial_0\tilde{f}(x)=\tilde{\alpha}\neq 0$$

or

$$\begin{aligned} f(\infty)&=-f(-\infty)=\tilde{\alpha}/2\\ \tilde{f}(\infty)&=-\tilde{f}(-\infty)=\alpha/2 \end{aligned} \tag{4.23}$$

then (4.21) defines a non-implementable automorphism of commutation relations. The translation (4.21) is formally generated by

$$U(f)=\exp\left(-\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x^1 f(x)\overleftrightarrow{\partial}_0\phi(x)\right) \tag{4.24}$$

which can be interpreted as an intertwining operator between the Fock representation and an inequivalent representation. The new vacuum is defined by

$$|\alpha,\tilde{\alpha}\rangle\equiv\exp\left(\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x^1 f(x)\overleftrightarrow{\partial}_0\phi(x)\right)|0'\} \tag{4.25}$$

and the Hilbert space $H_{\alpha,\tilde{\alpha}}$ is created from $|\alpha,\tilde{\alpha}\rangle$. It was shown by Ezawa (1979) that with conditions (4.22), Poincaré group transformations will be unitarily implemented on $H_{\alpha,\tilde{\alpha}}$. Hence we consider the representation $H_{\alpha,\tilde{\alpha}}$ as a physical one.

Let us choose the following example of the function $f(x)$ which is introduced into equation (4.24)

$$f(x)=\varepsilon(x^0-x^1). \tag{4.26}$$

Since $f(+\infty)=-1$ and $f(+\infty)+f(-\infty)=0$, then $\tilde{\alpha}=-2$. Function $\tilde{f}(x)$ is related to

$f(x)$ through

$$\tilde{f}(x)=\tfrac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}y^1\ \epsilon(x^1-y^1)\partial_0 f(x)=\varepsilon(x^1-x^0)=-f(x). \tag{4.27}$$

Hence $\alpha=2$ and

$$\int_{-\infty}^{\infty}\mathrm{d}x^1\ f(x)\overleftrightarrow{\partial}_0\phi(x)=2\phi(x^0,x^0)-\int_{-\infty}^{\infty}\mathrm{d}x^1\ \varepsilon(x^0-x^1)\partial_0\phi(x). \tag{4.28}$$

But as explained in § 3, the commutator

$$\left[\tilde{\phi}^{(\pm)}(y),\int_{-\infty}^{\infty}\mathrm{d}x^1\ \varepsilon(x^0-x^1)\partial_0\phi(x)\right]$$

is ill defined and (4.21) breaks down owing to these infrared divergencies. But it still has meaning to consider c-number translation of local operators $\phi^{(\pm)}(f)$ and $\tilde{\phi}^{(\pm)}(f)$ generated by (4.24). If $f(x)$ is as in (4.26) we obtain

$$\phi(g)\rightarrow\phi(g)+2g(x^0,x^0)-2\tilde{g}(x^0,x^0) \tag{4.29}$$

$$\tilde{\phi}(g)\rightarrow\tilde{\phi}(g)+2\tilde{g}(x^0,x^0)-2g(x^0,x^0) \tag{4.30}$$

where $g(x)$ is the test function satisfying condition $\int_{-\infty}^{\infty}\mathrm{d}x^1\ \partial_\mu g(x)=0$.

With the intention of circumventing these infrared difficulties described above, we introduce the intertwining operator

$$U(a,b)=\exp(-\mathrm{i}a\phi(u)-\mathrm{i}b\tilde{\phi}(u)) \tag{4.31}$$

where the argument u is held fixed. $U(a,b)$ generates the following c-number translation of local field operators

$$\phi(g)\rightarrow\phi(g)+ag(u)+b\tilde{g}(u) \tag{4.32}$$

$$\tilde{\phi}(g)\rightarrow(g)+a\tilde{g}(u)+bg(u). \tag{4.33}$$

We see that $U(a,b)$ is a generalisation of the global gauge transformation generator

$$U=\exp(-\mathrm{i}a\Phi-\mathrm{i}b\tilde{\Phi}) \tag{4.34}$$

which generates (4.19) and (4.20).

The operator $U(a,b)$ can now be used to introduce the vacuum of $H_{\alpha,\tilde{\alpha}}$ in an alternative way,

$$|\alpha,\tilde{\alpha}\rangle=U(\alpha,\tilde{\alpha})|0'\}. \tag{4.35}$$

The charges Φ and $\tilde{\Phi}$ are no longer equal to zero on $H_{\alpha,\tilde{\alpha}}$. From definition (4.23) we can easily evaluate

$$\Phi|\alpha,\tilde{\alpha}>=\int_{-\infty}^{\infty}\mathrm{d}x^1\ \partial_0 f(x)|\alpha,\tilde{\alpha}\rangle=\alpha|\alpha,\tilde{\alpha}\rangle \tag{4.36}$$

$$\begin{aligned}\tilde{\Phi}|\alpha,\tilde{\alpha}\rangle&=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}y^1\ \partial_0^y\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x^1\ f(x)\overleftrightarrow{\partial}_0[\tilde{\phi}(y),\phi(x)]|\alpha,\tilde{\alpha}\rangle\\&=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}y^1\ \partial_0^y\int\mathrm{d}x^1\ f(x)\overleftrightarrow{\partial}_0\tilde{D}(x-y)|\alpha,\tilde{\alpha}\rangle\\&=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}y^1\ \partial_0^y\tilde{f}(y)|\alpha,\tilde{\alpha}\rangle=\tilde{\alpha}|\alpha,\tilde{\alpha}\rangle.\end{aligned} \tag{4.37}$$

Of course, (4.36) and (4.37) can also be computed with the aid of equation (4.35). Then the intertwining operator $U(f)$ can be interpreted as a charge raising operator and inequivalent representations $H_{\alpha,\tilde{\alpha}}$ as different charge sectors. The charges Φ and $\tilde{\Phi}$ act on $H_{\alpha,\tilde{\alpha}}$ as the numbers α and $\tilde{\alpha}$ respectively.

5. The massless Thirring model

In this section we reconstruct the Thirring field from the massless Bose field. It was shown (Nakanishi 1977c) that the solution of the massless quantum Thirring model can be expressed in terms of $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$ in the following way

$$\psi(x) = \exp(-ib\gamma^5\tilde{\phi}^{(-)}(x))\exp(ia\phi^{(-)}(x))\exp(ia\phi^{(+)}(x))\exp(-ib\gamma^5\tilde{\phi}^{(+)}(x))u \tag{5.1}$$

where $u = \binom{u_1}{u_2}$ is a two-component c number. The field equation of the Thirring model is

$$i\gamma^\mu\partial_\mu\psi(x) = -g\gamma^\mu : J_\mu(x)\psi(x): \tag{5.2}$$

where we adopt Johnson's definition (Johnson 1961) of the current $J\mu(x)$:

$$J_\mu(x) = \tfrac{1}{2}[j_\mu(x) + \tilde{j}_\mu(x)] \tag{5.3}$$

$j_\mu(x)$ and $\tilde{j}_\mu(x)$ are given by

$$j_\mu(x) = \lim_{\substack{\varepsilon^0=0\\ \varepsilon^1\to 0}} j_\mu(x,\varepsilon)$$

$$= \lim_{\substack{\varepsilon^0=0\\ \varepsilon^2\to 0}} (-\varepsilon^2)^{(a^2+b^2)/4\pi-\frac{1}{2}} \sum_{r=1}^{2} \tfrac{1}{2}\{[\bar{\psi}(x+\varepsilon)\gamma_\mu]_r\psi_r(x) - \psi_r(x)[\bar{\psi}(x-\varepsilon)\gamma_\mu]_r\} \tag{5.4}$$

$$\tilde{j}_\mu(x) = \lim_{\substack{\varepsilon^0=0\\ \varepsilon^1\to 0}} j_\mu(x,\tilde{\varepsilon}) \tag{5.5}$$

where

$$\tilde{\varepsilon}^2 = -\varepsilon^2 \qquad \tilde{\varepsilon}\cdot\varepsilon = 0.$$

Before we show that $\psi(x)$ from equation (5.1) is the solution of equations (5.2), let us study the Lorentz transformation properties and statistics of the field $\psi(x)$. The Lorentz properties of the Fermi field $\psi(x)$ depend totally on the Lorentz properties of the Bose constituents $\phi^{(\pm)}(x)$ and $\tilde{\phi}^{(\pm)}(x)$ presented in (3.49)–(3.50).

Under the Lorentz transformation (with a boosting angle χ) $\psi(x)$ transforms as

$$\psi'(x') = \exp(-ib\gamma^5\tilde{\phi}^{(-)}(x) - b\chi\gamma^5\Phi/2\pi)\exp(ia\phi^{(-)}(x) + a\chi\tilde{\Phi}/2\pi)$$
$$\times\exp(ia\phi^{(+)}(x) - a\chi\tilde{\Phi}/2\pi)\exp(-ib\gamma^5\tilde{\phi}^{(+)}(x) + b\chi\gamma^5\Phi/2\pi)u. \tag{5.6}$$

The extra factor arises when Φ is moved to the middle of the second and third factors

$$\exp\left\{\left[-\frac{\chi b\gamma^5\Phi}{2\pi}, ia\phi^{(-)}(x)\right] + \left[ia\phi^{(+)}(x), \frac{\chi b\gamma^5\Phi}{2\pi}\right]\right\} = \exp\left(-\chi\frac{ab}{2\pi}\gamma^5\right). \tag{5.7}$$

The well known transformation property

$$\psi'(x') = \exp[-(\chi/2)\gamma^5]\psi(x) \tag{5.8}$$

is obtained when

$$ab = \pi. \tag{5.9}$$

From the commutation relations presented in § 3 we obtain

$$\psi_r(x)\psi_s(y) = \exp[-M_{rs}(x-y)]\psi_s(y)\psi_r(x) \tag{5.10}$$

$$\psi_r(x)\psi_s^*(y) = \exp[M_{rs}(x-y)]\psi_s^*(y)\psi_r(x) \tag{5.11}$$

where

$$M_{rs}(x-y) = \mathrm{i}[a^2 + b^2(-1)^{r+s}]D(x-y) - \mathrm{i}ab[(-1)^r + (-1)^s]\tilde{D}(x-y) \tag{5.12}$$

Then, since for $(x-y)^2 < 0$ we have $D(x-y) = 0$ and $\tilde{D}(x-y) = -\frac{1}{2}\varepsilon(x^1 - y^1)$, we find with $ab = (2n+1)\pi$, (n being an integer)

$$\begin{aligned} &\exp[\mp M_{rs}(x-y)] = -1 \qquad \text{for } (x-y)^2 < 0 \text{ and } r = s \\ &\exp[\mp M_{rs}(x-y)] = 1 \qquad \text{for } (x-y)^2 < 0 \text{ and } r \neq s. \end{aligned} \tag{5.13}$$

We conclude that for $s = \frac{1}{2}(ab = \pi)$ one gets Fermi statistics for the same spinor component but Bose statistics for different ones. However, a simple Klein transformation

$$\psi(x) \to \psi_K(x) = \exp(-\mathrm{i}a\,\tilde{\Phi}/2)\psi(x) \tag{5.14}$$

will remove such an anomaly and restore complete Fermi statistics (Stoyanov 1978, Hadjiivanov and Stoyanov 1979b). Now we calculate the current $J_\mu(x)$. We substitute (5.1) into (5.4) and (5.5), putting

$$|u_i|^2 = \mu^{(a^2+b^2)/2\pi}/2\pi \qquad i = 1, 2. \tag{5.15}$$

Taking the limits we obtain

$$J_\mu(x) = -\frac{a+b}{2\pi}\,\partial_\mu\phi(x) \tag{5.16}$$

for $ab = \pi$. Hence the field equation (5.2) is satisfied by $\psi(x)$ if and only if

$$b - a = \frac{g}{2\pi}(a+b). \tag{5.17}$$

Together with $ab = \pi$, equation (5.17) leads to

$$a^2 = \pi\frac{2\pi - g}{2\pi + g} \qquad b^2 = \pi\frac{2\pi + g}{2\pi - g} \tag{5.18}$$

under the assumption that $|g| < 2\pi$.

Now we consider the dilatation transformation of $\psi(x)$. Inserting (3.44) and the same relation with tildes in the definition (5.1) of $\psi(x)$ we obtain

$$\begin{aligned} \psi'(x) &= \exp(\mathrm{i}b\gamma^5\tilde{\phi}^{(-)}(\lambda x) - b\gamma^5\tilde{\Phi}\ln\lambda/2\pi)\exp(\mathrm{i}a\phi^{(-)}(\lambda x) + a\Phi\ln\lambda/2\pi) \\ &\quad\times\exp(\mathrm{i}a\phi^{(+)}(\lambda x) - a\Phi\ln\lambda/2\pi)\exp(-\mathrm{i}b\gamma^5\tilde{\phi}^{(+)}(\lambda x) + b\gamma^5\tilde{\Phi}\ln\lambda/2\pi)u \\ &= \psi(\lambda x)\exp\left(\frac{a^2+b^2}{4\pi}\ln\lambda\right) = \lambda^{(a^2+b^2)/4\pi}\psi(\lambda x) \end{aligned} \tag{5.19}$$

or commutating $\psi(x)$ directly with the generator D of scale transformations we obtain

$$[D, \psi(x)] = \mathrm{i}\left(x^\mu \partial_\mu + \frac{a^2+b^2}{4\pi}\right)\psi(x). \tag{5.20}$$

It is now obvious that we can attribute a definite scale dimension d_a to the field $\psi(x)$. From (5.19) we find

$$d_a = \frac{a^2+b^2}{4\pi} = \frac{1}{2}\frac{4\pi^2+g^2}{4\pi^2-g^2} = \frac{1}{2} + \frac{g^2}{4\pi^2-g^2} \tag{5.21}$$

where we used equation (5.18) with $|g|<2\pi$. For the non-interacting case we get $d_a = s = \frac{1}{2}$, as expected. Let us construct from $\psi(x)$ a composite field

$$\sigma_\pm(x) = :\bar{\psi}(x)(1\pm\gamma^5)\psi(x): \tag{5.22}$$

where the normal product $:\ldots:$ is defined by a space–time limiting procedure in the Thirring model. We see that

$$\begin{aligned}\sigma_+(x) &= 2:\psi_1^*(x)\psi_2(x): \\ \sigma_-(x) &= 2:\psi_2^*(x)\psi_1(x):.\end{aligned} \tag{5.23}$$

First we recall the well known formula

$$:\mathrm{e}^A::\mathrm{e}^B: = \mathrm{e}^{[A^+,B^-]}:\mathrm{e}^{A+B}:. \tag{5.24}$$

Hence

$$\begin{aligned}\psi_1^*(x+\varepsilon)\psi_2(x) &= \exp[(a^2-b^2)D^{(+)}(\varepsilon) - ab\tilde{D}^{(+)}(\varepsilon) + ab\tilde{D}^{(+)}(\varepsilon)] \\ &\quad\times :\exp[-\mathrm{i}b(\tilde{\phi}(x+\varepsilon)+\tilde{\phi}(x)) + \mathrm{i}a(\phi(x)-\phi(x+\varepsilon)]:u_1^*u_2 \\ &= (-\mu^2\varepsilon^2 - \mathrm{i}0\varepsilon^0)^{(b^2-a^2)/4\pi}:\exp[-\mathrm{i}b(\tilde{\phi}(x+\varepsilon) \\ &\quad+\tilde{\phi}(x)) + \mathrm{i}a(\phi(x)-\phi(x+\varepsilon)]:u_1^*u_2.\end{aligned} \tag{5.25}$$

We may now go one step further and write (5.23) in the form

$$\sigma_+(x) = \lim_{\varepsilon\to 0} N(\varepsilon)(\psi_1^*(x+\varepsilon)\psi_2(x) + \psi_2(x)\psi_1^*(x-\varepsilon)) \tag{5.26}$$

where

$$N(\varepsilon) = (-\mu^2\varepsilon^2 - \mathrm{i}0\varepsilon^0)^{(a^2-b^2)/4\pi}(u_1^*u_2)^{-1} \tag{5.27}$$

(here the limit can be taken without averaging with $\tilde{\varepsilon}$). Inserting (5.25) in (5.26) we obtain the final expression for $\sigma_\pm(x)$:

$$\sigma_\pm(x) = 2\exp(\mp 2\mathrm{i}b\tilde{\phi}^{(-)}(x))\exp(\mp 2\mathrm{i}b\tilde{\phi}^{(+)}(x)). \tag{5.28}$$

Under a scale transformation $\sigma_+(x)$ transforms as

$$\begin{aligned}\sigma'_+(x) &= 2\exp(-2\mathrm{i}b\tilde{\phi}^{(-)}(\lambda x) - b\tilde{\Phi}\ln\lambda/\pi)\exp(-2\mathrm{i}b\tilde{\phi}^{(+)}(\lambda x) + b\tilde{\Phi}\ln\lambda/\pi) \\ &= \exp(b^2\ln\lambda/\pi)\sigma_+(\lambda x) = \lambda^{b^2/\pi}\sigma_+(\lambda x).\end{aligned} \tag{5.29}$$

Hence we can assign an anomalous dimension to the fields $\sigma_\pm(x)$

$$d_{\sigma_+} = d_{\sigma_-} = \frac{b^2}{\pi} = \frac{2\pi+g}{2\pi-g}. \tag{5.30}$$

If we started with the field equation $\mathrm{i}\partial\!\!\!/\psi = g:J\!\!\!/\psi:$ we would obtain $d_{\sigma_\pm} =$

$(2\pi-g)/(2\pi+g)$. From (5.28) we find the both $\bar{\psi}(x)\psi(y)$ and $\bar{\psi}(x)\gamma^5\psi(y)$ are Lorentz invariant.

Let us now return to equation (4.8) and take the vacuum expectation value of the left- and right-hand sides:

$$\{'0|\psi'(x')|0'\} = \exp(-\chi\gamma^5/2)\{'0|\psi(x)|0'\}. \tag{5.31}$$

Inserting equation (5.6) in the left-hand side of equation (5.31) we find

$$\{'0|\psi'(x')|0'\} = \{'0|\psi(x)|0'\}. \tag{5.32}$$

We removed the charges Φ and $\tilde{\Phi}$ from the first and second factor appearing in (5.6) to the left where they act on the vacuum, and the charges from the third and fourth factor to the right. We used, of course, the fact that Φ and $\tilde{\Phi}$ annihilate the vacuum $|0'\}$. Hence (5.31) and (5.32) imply that

$$\{'0|\psi(x)|0'\} = 0. \tag{5.33}$$

Equation (5.33) is clear, because $\psi(x)$ carries the charge and $|0'\}$ defines the vacuum for the charge zero sector $H_{0,0}$. From definition (5.1) we find

$$[\Phi, \psi_r(x)] = a\psi_r(x) \tag{5.34}$$

$$[\tilde{\Phi}, \psi_r(x)] = -b(-1)^r\psi_r(x). \tag{5.35}$$

Hence

$$\Phi\psi_r(x)|0'\} = a\psi_r(x)|0'\} \tag{5.36}$$

$$\tilde{\Phi}\psi_r(x)|0'\} = -b(-1)^r\psi_r(x)|0'\} \tag{5.37}$$

and it follows that

$$\psi_r(x)|0'\} \in H_{a,-(-1)^r b}. \tag{5.38}$$

Accordingly

$$\langle\alpha, \tilde{\alpha}|\psi_r(x)|0'\} = 0 \qquad \text{when } (\alpha, \tilde{\alpha}) \neq (a, -(-1)^r b). \tag{5.39}$$

We see that the Thirring field acts as an intertwining operator from one charge sector to another.

Let us now consider the only non-vanishing matrix element

$$\langle a, -(-1)^r b|\psi_r'(x')|0'\}.$$

We arrange the charges appearing in (5.6) in such a way that the first Φ acts to the left and the rest act to the right. We obtain

$$\begin{aligned}\langle a, -(-1)^r b|&\exp[-b\chi(-1)^r\Phi/2\pi]\psi_r(x)|0'\} \\ &= \exp(-ab\chi(-1)^r/2\pi)\langle a, -(-1)^r b|\psi_r(x)|0'\}\end{aligned} \tag{5.40}$$

which is consistent with the spinor transformation law (5.8). Let us study dilatation transformations in the same way:

$$\begin{aligned}\langle a, -(-1)^r b|\psi_r'(x)|0'\} &\\ = \langle a, -(-1)^r b|&\exp[-ib(-1)^r\tilde{\phi}^{(-)}(\lambda x) - b(-1)^r\tilde{\Phi}\ln\lambda/2\pi] \\ &\times\exp[ia\phi^{(-)}(\lambda x) + a\Phi\ln\lambda/2\pi]\exp[ia\phi^{(+)}(\lambda x) - a\Phi\ln\lambda/2\pi] \\ &\times\exp[-ib(-1)^r\tilde{\phi}^{(+)}(\lambda x) + b(-1)^r\tilde{\Phi}\ln\lambda/2\pi]|0'\}.\end{aligned}$$

We get the factor $\exp(a^2 \ln \lambda/4\pi)$ from charges Φ when we move them into the middle of the second and third factors. Next we remove the first $\tilde{\Phi}$ to the left and the fourth to the right. This gives us the factor $\exp(-b^2 \ln \lambda/4\pi)$.

$$\begin{aligned}\langle a, -(-1)^r b|\psi'_r(x)|0'\} &= \exp\{[(a^2-b^2)/4\pi]\ln\lambda\}\langle a, -(-1)^r b|\exp[-b\tilde{\Phi}(-1)^r \ln\lambda/2\pi] \\ &\quad \times \psi_r(\lambda x)\exp[b(-1)^r\tilde{\Phi}\ln\lambda/2\pi]|0'\} \\ &= \exp[(a^2+b^2)/4\pi]\ln\lambda\}\langle a, -(-1)^r b|\psi_r(\lambda x)|0'\}. \end{aligned} \tag{5.41}$$

The result is consistent with the transformation law (5.19). It is easy to see that an arbitrary rearrangement of charges Φ, $\tilde{\Phi}$ between $\langle a, -(-1)^r b|$ and $|0'\}$ leads to (5.40) and (5.41).

It has been shown by Nakanishi (1977c) that the Wightman function

$$W_I(x, y) = \langle 0|\prod_{j=1}^{n} \psi^*_{r_j}(x_j) \prod_{k=1}^{m} \psi_{s_k}(y_k)|0\rangle$$

(where $x = \{x_1, \ldots, x_n\}$; $y = \{y_1, \ldots, y_m\}$ and $I = \{r_1, \ldots, r_n, s_1, \ldots, s_m\}$) in the limit $\mu \to 0$ satisfies the positivity condition, and

$$\lim_{\mu\to 0} W_I(x, y) = 0 \qquad \text{unless } n = m,\ p = q \tag{5.42}$$

where p and q are the number of j such that $r_j = 1$ and the number of k such that $s_k = 1$. For the definition of $|0\rangle$, see (6.20).

The question is whether there is any vacuum state which generates the functions $\lim_{\mu\to 0} W_I(x, y)$. We claim that $|0'\}$ is the right candidate. Let us consider

$$W'_I(x, y) = \{'0|\prod_{j=1}^{n} \psi^*_{r_j}(x_j) \prod_{k=1}^{m} \psi_{s_k}(y_k)|0'\}. \tag{5.43}$$

From the definition of $|0'\}$ as a vacuum of charge zero sector we easily find that

$$W'_I(x, y) = 0 \qquad \text{unless } n = m,\ p = q. \tag{5.44}$$

Let us illustrate the proof by an easy example. We choose the following non-vanishing function

$$\{'0|\psi^*_1(y)\psi_1(x)|0'\} \tag{5.45}$$

and using the Baker–Hausdorff formula we find

$$\begin{aligned}\{'0|\psi^*_1(y)\psi_1(x)|0'\} &= |u_1|^2 \exp[(a^2+b^2)D^{(+)}(y-x) + 2ab\tilde{D}^{(+)}(y-x)]F(x, y) \\ &= |u_1|^2(-\mu^2 z^2 + \mathrm{i}0z^0)^{-(a^2+b^2)/4\pi}\left(\frac{z^0 - z^1 - \mathrm{i}0}{z^0 + z^1 - \mathrm{i}0}\right)^{ab/2\pi} F(x, y)\end{aligned} \tag{5.46}$$

where $z = y - x$ and

$$\begin{aligned}F(x, y) = \{'0|&\exp[\mathrm{i}b(\tilde{\phi}^{(-)}(x) - \tilde{\phi}^{(-)}(y))]\exp[\mathrm{i}a(\phi^{(-)}(x) - \phi^{(-)}(y))] \\ &\times \exp[\mathrm{i}a(\phi^{(+)}(x) - \phi^{(+)}(y))]\exp[\mathrm{i}b(\tilde{\phi}^{(+)}(x) - \tilde{\phi}^{(+)}(y))]|0'\}\end{aligned} \tag{5.47}$$

By making use of the transformation formulae (3.49)–(3.52) we conclude that $F(x, y)$ is independent of its arguments. Setting $x = y$ we obtain $F(x, y) = 1$.

From (5.15) we obtain that (5.46) is independent of μ. By direct calculation we find

$$\{'0|\psi_1^*(y)\psi_1(x)|0'\} = \lim_{\mu\to 0} \langle 0|\psi_1^*(y)\psi_1(x)|0\rangle. \tag{5.48}$$

We see that the proof can be easily extended to the general case and that the Wightman functions coincide with the ones obtained in Klaiber's paper (1967).

6. Comments on Nakanishi's and Hadjiivanov and Stoyanov's papers

Nakanishi (1980) use the Fock representation in an indefinite-metric space, where the vacuum is defined by

$$\phi^{(+)}(x)|0\rangle = 0 \qquad \langle 0|0\rangle = 1 \tag{6.1}$$

hence

$$\langle 0|\phi(x)\phi(y)|0\rangle = D^{(+)}(x-y). \tag{6.2}$$

He assumed the following asymptotic conditions for $\partial_0\phi(x)$ and $\partial_1\phi(x)$

$$\partial_0\phi(x) \sim (x^1)^{-2} \tag{6.3}$$

$$\partial_1\phi(x) \sim (x^1)^{-1} \tag{6.4}$$

as $|x^1| \to \infty$.

The conjugate field $\tilde{\phi}(x)$ is defined in Nakanishi's framework by

$$\tilde{\phi}(x) = \int_{-\infty}^{x_1} \mathrm{d}y^1\, \partial_0\phi(x^0, y^1). \tag{6.5}$$

It is important to notice that only $\phi(x)$ is regarded by Nakanishi as a fundamental quantity, and therefore all properties of $\tilde{\phi}(x)$ should de derived from those of $\phi(x)$ (Nakanishi 1980). The conjugate field $\tilde{\phi}(x)$ satisfies

$$\Box\tilde{\phi}(x) = 0 \tag{6.6}$$

$$[\tilde{\phi}(x), \tilde{\phi}(y)] = \mathrm{i}D(x-y). \tag{6.7}$$

Equation (6.6) leads to the following definition of a new 'conjugate' charge' $\tilde{\Phi}$

$$\tilde{\Phi} = \int_{-\infty}^{\infty} \mathrm{d}x^1\, \partial_0\tilde{\phi}(x) = \int_{-\infty}^{\infty} \mathrm{d}x^1\, \partial_1\phi(x). \tag{6.8}$$

Because $\tilde{\Phi}$ commutes with $\phi(x)$ and $\langle 0|\tilde{\Phi}|0\rangle = 0$ we must put $\tilde{\Phi} = 0$ owing to the assumed irreducibility of the field $\phi(x)$. On the other hand the asymptotic behaviour of $\partial_1\phi(x)$ shown in (6.4) implies that $\tilde{\Phi}$ is ill defined.

The ambiguous status of $\tilde{\Phi}$ in Nakanishi's formalism reveals, as we shall see later on, an important discrepancy between the asymptotic behaviour of $\phi(x)$ and the irreducibility of ϕ.

Since the symmetry operation $\tilde{\phi}(x) \to \tilde{\phi}(x) + c$ cannot be introduced on the basis of the definition (6.5), so the chiral symmetry of the Fermi field $\psi(x) \to \exp(ic\gamma^5)\psi(x)$ cannot be introduced in this formalism either. And this is quite reasonable because $c\tilde{\Phi}$ would be a generator of such a symmetry. In conclusion we note that we have a rather strange situation with the two different conserved currents $\partial_\mu\phi(x)$ and $\partial_\mu\tilde{\phi}(x)$ but only one well defined charge. The dilatation generator is defined by

$$D \equiv \int_{-\infty}^{\infty} dx^1 \{x_1 \mathscr{T}_{10}(x) - x_0 \mathscr{T}_{00}(x)\} \tag{6.9}$$

where

$$\mathscr{T}_{00} \equiv \partial_0 \phi^{(-)} \partial_0 \phi^{(+)} + \partial_1 \phi^{-} \partial_1 \phi^{(+)} \tag{6.10}$$

$$\mathscr{T}_{10} \equiv \partial_0 \phi^{(-)} \partial_1 \phi^{(+)} + \partial_1 \phi^{(-)} \partial_0 \phi^{(+)}. \tag{6.11}$$

Thanks to conditions (6.3) and (6.4) the integral in (6.9) is convergent and the generator D exists and is well defined.

It can be shown in the standard way (see the derivation of the formula (3.49) that

$$[\phi^{(\pm)}(x), D] = -\mathrm{i} x^{\mu} \partial_{\mu} \phi^{(\pm)}(x) \pm \Phi^{(\pm)}/2\pi. \tag{6.12}$$

Thus we can write

$$\phi^{(\pm)\prime}(x) = \phi^{(\pm)}(\lambda x) \pm \mathrm{i} \ln \lambda \, \Phi^{(\pm)}/2\pi. \tag{6.13}$$

From definition (6.1) of the vacuum state we see that the vacuum is invariant under scale transformations (6.13). It is straightforward to see that the transformation formula (6.13) leaves the commutators of $\phi^{(\pm)}(x)$ invariant.

Let us now assume that $\tilde{\phi}(x)$ transforms under scale transformation as

$$\tilde{\phi}'(x) = \tilde{\phi}(\lambda x) + X \tag{6.14}$$

where X is some unknown term. Applying the scale transformation to the commutator

$$[\phi^{(\pm)}(x), \tilde{\phi}(y)] = \tilde{D}^{(\pm)}(x-y) \tag{6.15}$$

we obtain the following relation

$$[\phi^{(\pm)}(\lambda x), X] = \mp \frac{\mathrm{i} \ln \lambda}{2\pi} [\Phi^{(\pm)}, X]. \tag{6.16}$$

But because the commutator $[\Phi^{(\pm)}, \tilde{\phi}(x)] = 0$ must be invariant under scale transformations we find that X must obey

$$[\phi^{(\pm)}(\lambda x), X] = 0. \tag{6.17}$$

Therefore the extra term X must vanish because of the irreducibility of $\phi(x)$ and $\langle 0|X|0\rangle = 0$. The same considerations can be applied to $\tilde{\phi}^{(\pm)}$ transformation laws with the result that $\tilde{\phi}^{(\pm)}(x)$ are invariant under scale transformation. But then we get contradiction with

$$[\tilde{\phi}^{(\pm)}(\lambda x), \tilde{\phi}^{(\mp)}(\lambda y)] = D^{(\pm)}(x-y) \pm \ln \lambda/2\pi \neq D^{(\pm)}(x-y). \tag{6.18}$$

The commutator between $\tilde{\phi}^{(\pm)}(x)$ and the scale transformation generator D can easily be calculated in Nakanishi's approach in the following way

$$\begin{aligned}
[\tilde{\phi}^{(\pm)}(x), D] &= \lim_{L\to\infty} \lim_{M\to\infty} \left(-\mathrm{i} \int_{-L}^{L} dy^1 \, D^{(\pm)}(x-y) \tilde{\partial}_0 \int_{-M}^{y^1} du^1 \, \partial_0 \phi(y^0, u^1), D \right) \\
&= -\lim_{L\to\infty} \lim_{M\to\infty} \int_{-L}^{L} dy^1 \, D^{(\pm)}(x-y) \tilde{\partial}_0 \int_{-M}^{y^1} du^1 \, \partial_0 (y_0 \partial_0 - u_1 \partial_1) \phi(y^0, u^1) \\
&= -\mathrm{i} x^{\mu} \partial_{\mu} \tilde{\phi}^{(\pm)}(x) \pm \tilde{\Phi}/2\pi.
\end{aligned} \tag{6.19}$$

We can now recognise that the troubles with scale transformation are equivalent to the troubles with $\tilde{\Phi}$ in Nakanishi's approach. If $\tilde{\Phi}$ is well defined, then we must put $\tilde{\Phi}=0$ and claim $\tilde{\phi}^{(\pm)}(x)$ to be invariant under scale transformation which leads us to the contradiction shown in (6.18). If $\tilde{\Phi}$ is ill defined, then, according to (6.19) and the fact that D is well defined, $\tilde{\phi}^{(\pm)}(x)$ must be ill defined too. Since $\partial_0\tilde{\phi}(x)$ behaves like $(x^1)^{-1}$, see (6.4), the last remark is easy to accept remembering that $\tilde{\phi}^{(\pm)}(x) = -i\int_{-\infty}^{\infty} dy^1\, D^{(\pm)}(x-y)\overleftrightarrow{\partial}_0^y\phi(y)$. Anyway, we cannot introduce the scale transformation in the Thirring model with Nakanishi's formalism although the gauge-invariant Wightman functions corresponding to the operator solution of the Thirring model exhibit scale invariance.

The last remark concerns the existence of Goldstone bosons. Using $[\Phi, \phi(x)] = -i$ it is concluded by Nakanishi that $\langle 0|[\Phi, \phi(x)]|0\rangle = -i$ and therefore that $\Phi|0\rangle \neq 0$. In our construction it is clear that Φ (as well as $\tilde{\Phi}$) is well defined by superselection rules and it does annihilate the vacuum. The troubles come from the from the fact that $\phi(x)$ is an illegal operator and must be smeared with appropriate test functions when we require the physical space to be a positive definite Hilbert space.

In Hadjiivanov *et al* framework (1979) the vacuum is introduced by

$$\phi^{(+)}(x)|0\rangle = \tilde{\phi}^{(+)}(x)|0\rangle = 0 \tag{6.20}$$

and the second charge $\tilde{\Phi}$ exists.

The transformation properties of $\phi^{(\pm)}(x)$, $\tilde{\phi}^{(\pm)}(x)$ under Poincaré and dilatation transformations are postulated by imposing the condition that both equation (6.20) and the commutation relations must be invariant under transformations. This leads (in our notation) to

$$\phi^{(\pm)\prime}(x') = \phi^{(\pm)}(x) \pm i\chi\tilde{\Phi}^{(\pm)}/2\pi \tag{6.21}$$

$$\tilde{\phi}^{(\pm)\prime}(x') = \tilde{\phi}^{(\pm)}(x) \pm i\chi\Phi^{(\pm)}/2\pi \tag{6.22}$$

$$\phi^{(\pm)\prime}(x) = \phi^{(\pm)}(\lambda x) \pm i\ln\lambda\,\Phi^{(\pm)}/2\pi \tag{6.23}$$

$$\tilde{\phi}^{(\pm)\prime}(x) = \tilde{\phi}^{(\pm)}(\lambda x) \pm i\ln\lambda\,\tilde{\Phi}^{(\pm)}/2\pi. \tag{6.24}$$

Of course the fields $\phi(x)$ and $\tilde{\phi}(x)$ will not be invariant, but transform inhomogeneously under transformations (6.21)–(6.24). This explains the fact that the right-hand side of equation

$$\langle 0|\phi(x)\tilde{\phi}(y)|0\rangle = \tilde{D}^{(+)}(x-y) \tag{6.25}$$

is not Lorentz invariant and the right-hand side of equation (6.2) is not invariant under scale transformations. In our approach the fields $\phi(x)$, $\tilde{\phi}(y)$ are scalars but the state $|0\rangle$ defined in (6.20) is spontaneously broken under Lorentz and scale transformations.

If U_λ is a scaling symmetry operator, then

$$0 = U_\lambda\phi^{(+)}(x)|0\rangle = U_\lambda\phi^{(+)}(x)U_\lambda^{-1}U_\lambda|0\rangle = (\phi^{(+)}(\lambda x) + i\ln\lambda\Phi/2\pi)|0_\lambda\rangle \tag{6.26}$$

where $U_\lambda|0\rangle = |0_\lambda\rangle \neq |0\rangle$ because $\Phi^{(-)}|0\rangle \neq 0$. Accordingly

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0_\lambda|\phi(\lambda x)\phi(\lambda y)|0_\lambda\rangle \tag{6.27}$$

Applying equation (6.26) we can show that the right-hand side of equation (6.27) is still equal to $D^{(+)}(x-y)$ and there is no contradiction. The state $|0\rangle$ is not the vacuum state for the physical Hilbert space with positive metric; thus this spontaneous symmetry breakdown has no significance.

The transformations (6.21)–(6.24) can be easily reproduced if we define normal ordered Poincaré and diltation generators by

$$M_{01}=\lim_{L\to\infty}\int_{-L}^{L}\mathrm{d}x^{1}(x_{0}\mathcal{T}_{10}(x)-x_{1}\mathcal{T}_{00}(x)) \tag{6.28}$$

and by (6.9). These generators annihilate $|0\rangle$ and a simple calculation (see the proof of (3.49)) leads to

$$[\phi^{(\pm)}(x),M_{01}]=\mathrm{i}(x_{0}\partial_{1}-x_{1}\partial_{0})\phi^{(\pm)}(x)\mp\tilde{\Phi}^{(\pm)}/2\pi \tag{6.29}$$

$$[\tilde{\phi}^{(\pm)}(x),M_{01}]=\mathrm{i}(x_{0}\partial_{1}-x_{1}\partial_{0})\tilde{\phi}^{(\pm)}(x)\mp\Phi^{(\pm)}/2\pi \tag{6.30}$$

$$[\phi^{(\pm)}(x),D]=-\mathrm{i}x^{\mu}\partial_{\mu}\phi^{(\mp)}(x)\pm\Phi^{(\pm)}/2\pi \tag{6.31}$$

$$[\tilde{\phi}^{(\pm)}(x),D]=-\mathrm{i}x^{\mu}\partial_{\mu}\tilde{\phi}^{(\pm)}(x)\pm\tilde{\Phi}^{(\pm)}/2\pi. \tag{6.32}$$

These equations are, of course, compatible with equations (6.21)–(6.24).

Inserting equations (6.21)–(6.24) into (5.1) we obtain

$$\begin{aligned}\psi'(x')=&\exp[-(b\chi/2\pi)\gamma^{5}\Phi^{(-)}]\exp[(a\chi/2\pi)\tilde{\Phi}^{(-)}]\psi(x)\\&\times\exp[-(a\chi)/2\pi)\tilde{\Phi}^{(+)}]\exp[(b\chi/2\pi)\gamma^{5}\Phi^{(+)}]\end{aligned} \tag{6.33}$$

and

$$\begin{aligned}\psi'(x)=&\exp[-(b\gamma^{5}\ln\lambda/2\pi)\tilde{\Phi}^{(-)}]\exp[(a\ln\lambda/2\pi)\Phi^{(-)}]\psi(\lambda x)\\&\times\exp[-(a\ln\lambda/2\pi)\tilde{\Phi}^{(+)}]\exp[(b\gamma^{5}\ln\lambda)/2\pi)\tilde{\Phi}^{(+)}].\end{aligned} \tag{6.34}$$

The transformation laws (6.33) and (6.34) are not the standard ones and therefore one cannot assign any spin or conformal dimension to the field $\psi(x)$. In §5 we saw that the field $\psi_r(x)$ is an intertwining operator between zero charge sector and $H_{a,-(-1)^r b}$. The charges Φ and $\tilde{\Phi}$ act on $H_{a,-(-1)^r b}$ as a numbers a and $-(-1)^r b$ respectively.

With this identification we find

$$\begin{aligned}\psi_r'(x')|0'\}=&\exp[-(b\chi/2\pi)(-1)^{r}\Phi^{(-)}]\exp[(a\chi/2\pi)\tilde{\Phi}^{(-)}]\psi_r(x)|0'\}\\&\times\exp[-(\chi ab(-1)^{r}/2\pi)]\psi_r(x)|0'\}\end{aligned} \tag{6.35}$$

$$\begin{aligned}\psi_r'(x)|0'\}&=\exp\{[(b(-1)^{r}/2\pi]\ln\tilde{\Phi}^{(-)}\}\exp[(a\ln\lambda/2\pi)\Phi^{(-)}]\psi_r(\lambda x)|0'\}\\&=\exp\{[(a^{2}+b^{2})/4\pi]\ln\lambda\}\psi_r(\lambda x)|0'\}.\end{aligned} \tag{6.36}$$

Thus, all non-vanishing matrix elements of $\psi(x)$ still transform with spin $\frac{1}{2}(ab=\pi)$ and anomalous dimension $d_a=\frac{1}{2}+g^{2}/(4\pi^{2}-g^{2})$. In general it is easy to verify that the gauge-invariant functions

$$\{'0|\bar{\psi}(x'_{2n})\dots\bar{\psi}(x'_{n+1})\psi(x'_{n})\dots\psi(x'_{1})|0'\}$$

transform under Lorentz transformations to

$$\begin{aligned}\{'0|\bar{\psi}(x_{2n})&\exp[-(bx/2\pi)\gamma^{5}_{2n}\Phi^{(+)}]\exp[(bx/2\pi)\gamma^{5}_{2n-1}\Phi^{(-)}]\bar{\psi}(x_{2n-1})\dots\bar{\psi}(x_{n+1})\\&\times\exp[(a\chi/2\pi)\tilde{\Phi}^{(+)}]\exp[-(b\chi/2\pi)\gamma^{5}_{n+1}\Phi^{(+)}]\exp[-(b\chi/2\pi)\gamma^{5}_{n}\Phi^{(-)}]\\&\times\exp[(a\chi/2\pi)\tilde{\Phi}^{(-)}]\psi(x_{n})\dots\psi(x_{2})\\&\times\exp[(b\chi/2\pi)\gamma^{5}_{2}\Phi^{(+)}]\exp[-(b\chi/2\pi)\gamma^{5}_{1}\Phi^{(-)}]\psi(x_{1})|0'\}\end{aligned}$$

$$= \exp\left(-\frac{ab}{2\pi}\chi \sum_{k=n+1}^{2n} \gamma_k^5\right)\{'0|\bar{\psi}(x_{2n}) \ldots \bar{\psi}(x_{n+1})$$

$$\times \psi(x_n) \ldots \psi(x_1)|0'\} \exp\left(-\frac{ab}{2\pi}\chi \sum_{L=1}^{n} \gamma_L^5\right) \tag{6.37}$$

and under the scale transformations

$$\{'0|\bar{\psi}(x_{2n}) \ldots \bar{\psi}(x_{n+1})\psi(x_n) \ldots \psi(x_1)|0'\}$$
$$= \lambda^{2n(a^2+b^2)/4\pi}\{'0|\bar{\psi}(\lambda x_{2n}) \ldots \bar{\psi}(\lambda x_{n+1})\psi(\lambda x_n) \ldots \psi(\lambda x_1)|0'\}. \tag{6.38}$$

The same phenomenon emerges when we consider Wightman functions

$$W_I(x, y) = \langle 0|\prod_j \bar{\psi}(x_j)\prod_k \psi(y_k)|0\rangle \tag{6.39}$$

defined on the state $|0\rangle$. In order to get the positive definite theory we let $\mu \to 0$ (μ is the regularisation term). In the limit $\mu \to 0$ in the Wightman functions the field $\psi(x)$ acquires the fixed spin (Hadjiivanov and Stoyanov 1979b)

$$s = ab/2\pi$$

and scale dimension

$$d = (a^2 + b^2)/4\pi.$$

In both theories there are no trace of spontaneous breaking of the gauge symmetries.

7. Conclusions

We have seen in this paper that it is possible to formulate a consistent quantum theory of a two-dimensional free massless scalar field $\phi(x)$ in a positive definite Hilbert space.

Many of the problems connected with bosonisation have been analysed in terms of the operators Φ and $\tilde{\Phi}$.

The fermionic selection rule emerged from charge superselection rules defined by these two operators. Our operator solution to the Thirring model possesses the following reasonable properties: Wightman functions are vanishing for non-equal numbers of ψ and ψ^*, gauge transformations of the first kind are trivially implementable on each charge sector, the spinor fields are transformed in the standard way both under Lorentz and scale transformations and, contrary to Nakanishi's framework, the chiral charge is well defined.

Remembering that the blame for all problems with the indefinite metric can be laid uniquely on the regularisation term μ, it is natural to expect that, in the limit $\mu \to 0$, the Wightman functions will define the positive two-dimensional theory.

The Wightman functions obtained in that limit have been shown to be identical with those generated from the vacuum of our charge zero sector.

The existence of the charge sectors in the massless boson field theory exhibiting fermionic degrees of freedom is closely related to the problem of attributing spin and scale dimension to the Thirring field $\psi(x)$; this explains, in an easy way, the results obtained by Hadjiivanov and Stoyanov (1979b) in the limit $\mu \to 0$.

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Appendix

We present here some facts and definitions about the functions $D^{(\pm)}(x)$ and $\tilde{D}^{(\pm)}(x)$. To introduce the positive—and negative—frequency parts of $D(x)$, it is necessary to use an infrared cut-off, as shown by Klaiber (1967). Following Nakanishi (1977b) we define $D^{(+)}(x)$ by

$$D^{(+)}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dp^1}{2p^0}[e^{-ipx}-\theta(\kappa-p^0)$$

$$=-\frac{1}{4\pi}\lg(-\mu^2x^2+i0x^0) \tag{A.1}$$

where $\mu=e^{\gamma}\kappa$, $\gamma=-\Gamma'(1)$ and κ is an arbitrary constant.

$D^{(-)}(x)$ is related to $D^{(+)}(x)$ by

$$D^{(-)}(x)=-D^{(+)}(-x)=-[D^{(+)}(x)]^* \tag{A.2}$$

$$iD(x)=D^{(-)}(x)+D^{(+)}(x) \tag{A.3}$$

and, of course,

$$\Box D^{(\pm)}(x)=0. \tag{A.4}$$

Now we introduce $\tilde{D}^{(+)}(x)$

$$\tilde{D}^{(+)}(x)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dp^1}{2p^1}[e^{-ipx}-\theta(\kappa-p^0)]$$

$$=\frac{1}{4\pi}\lg\left(\frac{x^0-x^1-i0}{x^0+x^1-i0}\right)=\frac{1}{4\pi}\left(\ln\left|\frac{x^0-x^1}{x^0+x^1}\right|-i\pi\varepsilon(x^1)\theta(-x^2)\right) \tag{A.5}$$

We define $\tilde{D}^{(-)}(x)$ in such a way that relations (A.2)–(A.4) with tildes hold. Some basic formulae are (Nakanishi 1977b)

$$\int_{-\infty}^{\infty}dz'\,D^{(\pm)}(x-z)\overleftrightarrow{\partial}_0^zD^{(\mp)}(z-y)=0 \tag{A.6}$$

$$\int_{-\infty}^{\infty}dz'\,D^{(\pm)}(x-z)\overleftrightarrow{\partial}_0^z\tilde{D}^{(\mp)}(z-y)=0. \tag{A.7}$$

From (2.4) with $f(x)=D^{(\pm)}(x)$, together with (2.8), we see that (A.6) and (A.7) imply

$$-i\int_{-\infty}^{\infty}dz'\,D^{(\pm)}(x-z)\overleftrightarrow{\partial}_0^zD^{(\pm)}(z-y)=D^{(\pm)}(x-y) \tag{A.8}$$

$$-i\int_{-\infty}^{\infty}dz'\,D^{(\pm)}(x-z)\overleftrightarrow{\partial}_0^z\tilde{D}^{(\pm)}(z-y)=\tilde{D}^{(\pm)}(x-y). \tag{A.9}$$

From the explicit expressions for $D^{(\pm)}(x)$ and $\tilde{D}^{(\pm)}(x)$ we obtain

$$\partial_0 D^{(\pm)}(x) = \partial_1 \tilde{D}^{(\pm)}(x) = \mp \frac{1}{2\pi} \frac{x^0}{x^2 \mp i0x^0} \tag{A.10}$$

$$\partial_1 D^{(\pm)}(x) = \partial_0 \tilde{D}^{(\pm)}(x) = \pm \frac{1}{2\pi} \frac{x^1}{x^2 \mp i0x^0}. \tag{A.11}$$

We have

$$\int_{-\infty}^{\infty} dx^1\, \partial_0 D^{(\pm)}(x) = \int_{-\infty}^{\infty} dx^1\, \partial_1 \tilde{D}^{(\pm)}(x) = -\frac{i}{2} \tag{A.12}$$

but $\int_{-\infty}^{\infty} dx^1\, \partial_0 \tilde{D}^{(\pm)}(x)$ does not exist because $D^{(\pm)}(x)$ is logarithmically divergent.

The function $\tilde{D}^{(\pm)}(x)$ is not invariant under Lorentz transformation, whereas the function $D^{(\pm)}(x)$ is not invariant under the scale transformation. Under the Lorentz transformation

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}$$

$\tilde{D}^{(\pm)}(x)$ transforms as

$$\tilde{D}^{(\pm)}(\Lambda^{-1}x) = \tilde{D}^{(\pm)}(x) \pm \chi/2\pi. \tag{A.13}$$

Under the scale transformation $x \to \lambda x$, $D^{(\pm)}(x)$ transforms as

$$D^{(\pm)}(\lambda x) = D^{(\pm)}(x) \mp \ln\lambda/2\pi. \tag{A.14}$$

Combining the equations (A.10) and (A.11) we have the following relations

$$(x_0\partial_0 - x_1\partial_1)D^{(\pm)}(x) = (x_0\partial_1 - x_1\partial_0)\tilde{D}^{(\pm)}(x) = \mp 1/2\pi \tag{A.15}$$

$$(x_0\partial_1 - x_1\partial_0)D^{(\pm)}(x) = (x_0\partial_0 - x_1\partial_1)\tilde{D}^{(\pm)}(x) = 0. \tag{A.16}$$

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